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On the existence of Burgers vortices for high Reynolds numbers ☆

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ABSTRACT

Axisymmetric or non-axisymmetric Burgers vortices have been studied numerically as a model of concentrated vorticity fields. Recently it has been rigorously proved that non-axisymmetric Burgers vortices exist for all values of the vortex Reynolds number if an asymmetry parameter is sufficiently small. On the other hand, several numerical results indicate that Burgers vortices have simpler structures as the vortex Reynolds number is increasing, even when the asymmetry parameter is not sufficiently small. In this paper we give a rigorous explanation for this numerical observation and extend the existence and stability results of Burgers vortices for high vortex Reynolds numbers.

1. Introduction

In 1948 J.M. Burgers [1] found an exact solution to the three-dimensional stationary Navier–Stokes equations for viscous incompressible fluids as follows. We consider a two-dimensional perturbation of a background straining flow whose velocity is of the form:

$$U(x_1, x_2, x_3) = u_\lambda(x_1, x_2, x_3) + u(x_1, x_2),$$

where u_λ represents a given background straining flow with an asymmetry parameter $\lambda \in [0, 1)$, and u is an unknown two-dimensional perturbation, i.e.,

$$u_\lambda(x_1, x_2, x_3) = \left(-\frac{1+\lambda}{2}x_1, -\frac{1-\lambda}{2}x_2, x_3\right),$$
$$u(x_1, x_2) = \big(u_1(x_1, x_2), u_2(x_1, x_2), 0\big)$$

with $\partial_1 u_1 + \partial_2 u_2 = 0$.

Taking rotation of the velocity U, we find that the vorticity vector has only one component depending only on two spatial variables:

$$\nabla \times U = \big(0, 0, \omega(x_1, x_2)\big)$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$. We assume that ω is integrable. The value $\alpha = \int_{\mathbb{R}^2} \omega(x)\,dx$ is called the total circulation, and $|\alpha|$ is called the (vortex) Reynolds number. Assuming that U satisfies the three-dimensional stationary Navier–Stokes equations for viscous incompressible fluids, we obtain the equations for ω as follows:

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$$\begin{cases} \mathcal{L}\omega = (u, \nabla)\omega - \lambda \mathcal{M}\omega, \quad x \in \mathbb{R}^2, \\ u = K * \omega, \\ \displaystyle\int_{\mathbb{R}^2} \omega(x)\, dx = \alpha. \end{cases} \tag{$B_{\lambda,\alpha}$}$$

Here the operators $\mathcal{L}$ and $\mathcal{M}$ are given by

$$\mathcal{L} = \Delta + \frac{x}{2} \cdot \nabla + 1, \qquad \mathcal{M} = \frac{1}{2}(x_1 \partial_1 - x_2 \partial_2). \tag{1.1}$$

The relation between the velocity field u and the vorticity ω is called the Biot–Savart law, and the convolution kernel K is given by

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad x^\perp = (-x_2, x_1). \tag{1.2}$$

The aim of this paper is to construct a solution to $(B_{\lambda,\alpha})$ under some assumptions for α and λ. We call a solution to $(B_{\lambda,\alpha})$ the Burgers vortex.

Let G be the two-dimensional Gauss kernel:

$$G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}. \tag{1.3}$$

Then by direct calculations, we see that G satisfies $\mathcal{L}G = 0$, $(K * G, \nabla)G = 0$. Thus αG solves Eq. $(B_{\lambda,\alpha})$ for $\lambda = 0$. This is the exact solution found by J.M. Burgers, and it is called the axisymmetric Burgers vortex. The stability of the axisymmetric Burgers vortices was firstly discussed by Y. Giga and T. Kambe [7] for small $|\alpha|$ (see also the related work by A. Carpio [2] and Y. Giga and M.-H. Giga [8]), and this smallness assumption was removed by Th. Gallay and C.E. Wayne [4].

The case $\lambda \neq 0$ is called non-axisymmetric. In this case $(B_{\lambda,\alpha})$ has not yet been studied much. As far as the author knows, the only mathematical results are the results by Th. Gallay and C.E. Wayne [5,6]. In [6] they constructed solutions to $(B_{\lambda,\alpha})$ in the Gaussian weighted L^2 space for any Reynolds number $|\alpha|$ when the asymmetry parameter λ is sufficiently small ($\lambda \ll \frac{1}{2}$). For not sufficiently small λ, the existence and uniqueness of the Burgers vortex are obtained in [5] only when the Reynolds number $|\alpha|$ is sufficiently small (the smallness of $|\alpha|$ depends on $\lambda \in [0, 1)$). Roughly speaking, when λ is not sufficiently small the term $\lambda \mathcal{M}\omega$ leads to the slow spatial decay in x_2 direction, which causes difficulties in controlling the nonlinear term.

The Burgers vortex, or Eq. $(B_{\lambda,\alpha})$, has been used as a model which describes local structures of intense vorticity fields in turbulence. Although there are only a few mathematical results, Burgers vortices have been well studied numerically; see A.C. Robinson and P.G. Saffman [16], S. Kida and K. Ohkitani [9], H.K. Moffatt, S. Kida, and K. Ohkitani [11], A. Prochazka and D.I. Pullin [14,15]. In these papers the case of high Reynolds numbers is mainly investigated from physical motivations. The interesting feature of their results is that the Burgers vortex has simpler structures and better stability when the Reynolds number $|\alpha|$ is large even for not small λ. Especially, it is numerically shown that the shape of the isovorticity contour becomes more circular as the Reynolds number is increasing; [9,15,16]. In [11] the asymptotic expansion of Burgers vortices at large α is formally obtained, which well expresses the simple structures of Burgers vortices for high Reynolds numbers. In the previous work [12] the author studied a linearized operator of $(B_{\lambda,\alpha})$ for $\lambda = 0$ and obtained some estimates and spectrum behavior for this operator which are compatible with the numerical results.

In the present paper we consider Eq. $(B_{\lambda,\alpha})$ when the Reynolds number $|\alpha|$ is sufficiently large, and the asymmetry parameter λ is less than $\frac{1}{2}$; as for the restriction $\lambda < \frac{1}{2}$, see Remark 1.3.

To state our results precisely, let us introduce function spaces. Let X, Y be the complex Hilbert spaces defined as follows:

$$X = \left\{ w \in L^2(\mathbb{R}^2) \;\Big|\; G^{-\frac{1}{2}} w \in L^2(\mathbb{R}^2),\ \int_{\mathbb{R}^2} w\, dx = 0,\ \langle w_1, w_2 \rangle_X = \int_{\mathbb{R}^2} G^{-1}(x) w_1(x) \overline{w_2(x)}\, dx \right\}, \tag{1.4}$$

$$Y = \left\{ w \in X \;\Big|\; \partial_i w \in X,\ i = 1, 2,\ \langle w_1, w_2 \rangle_Y = \int_{\mathbb{R}^2} G^{-1}(x)\big(w_1(x)\overline{w_2(x)} + \nabla w_1(x) \cdot \overline{\nabla w_2(x)}\big)\, dx \right\}. \tag{1.5}$$

We also define the subspace of X

$$W = \left\{ w \in X \;\Big|\; G^{-\frac{1}{2}} x_i w \in L^2(\mathbb{R}^2),\ i = 1, 2,\ \langle w_1, w_2 \rangle_W = \int_{\mathbb{R}^2} G^{-1}(x)\big(w_1(x)\overline{w_2(x)} + |x|^2 w_1(x)\overline{w_2(x)}\big)\, dx \right\}. \tag{1.6}$$

Clearly the closed subspace $Y \cap W$ (equipped with the natural scalar product) is compactly embedded in X. Motivated by [12], we set $\mathbb{P}_S X$ as the space of all radially symmetric functions in X, i.e.,

$$\mathbb{P}_S X = \big\{ f \in X \;\big|\; f(Rx) = f(x) \text{ a.e. } x \in \mathbb{R}^2 \text{ for all orthogonal matrix } R \big\}. \tag{1.7}$$

Let $\mathbb{P}_{S^\perp}X$ be the orthogonal complement of $\mathbb{P}_S X$ in X. Let $w_\infty \in Y \cap W \cap \mathbb{P}_{S^\perp}X$ be the real-valued function which satisfies the equation

$$\mathcal{M}G = \Lambda_G w_\infty, \tag{1.8}$$

where

$$\Lambda_G f = (K * G, \nabla) f + (K * f, \nabla) G. \tag{1.9}$$

The existence of w_∞ is obtained in [6] (see also [11]). In fact, w_∞ is uniquely determined in $\mathbb{P}_{S^\perp}X$; see Section 2. The function w_∞ is required when we consider the asymptotic expansion of Burgers vortices at large Reynolds numbers. Our main result is as follows.

Theorem 1.1 *(Existence). Let $\lambda \in [0, \frac{1}{2})$. Then there is a number $R(\lambda) \geqslant 0$ such that for any $\alpha \in \mathbb{R}$ with $|\alpha| \geqslant R(\lambda)$, there exists a (real-valued) solution $\omega_{\alpha,\lambda}$ to $(B_{\lambda,\alpha})$ such that $\omega_{\alpha,\lambda} - \alpha G \in Y \cap W$, $\int_{\mathbb{R}^2} x_i \omega_{\alpha,\lambda}\, dx = 0$, $i = 1, 2$, and*

$$\|\omega_{\alpha,\lambda} - \alpha G - \lambda w_\infty\|_{Y\cap W} \leqslant \frac{C\lambda}{(1-2\lambda)(1+|\alpha|)} \tag{1.10}$$

where the constant C is independent of α and λ. The quantity $R(\lambda)$ satisfies

$$\lim_{\lambda \to \frac{1}{2}} R(\lambda) = \infty. \tag{1.11}$$

Theorem 1.2 *(Uniqueness). Let $\lambda \in [0, \frac{1}{2})$. Then for any $\tau > 0$ there is a positive number $R'(\lambda, \tau)$ such that for any α with $|\alpha| \geqslant R'(\lambda, \tau)$, there exists at most one solution to $(B_{\lambda,\alpha})$ in the ball*

$$\mathcal{B}_\tau = \left\{ f \in L^2\left(\frac{1}{G}\, dx\right) \;\Big|\; \int_{\mathbb{R}^2} x_i f(x)\, dx = 0,\ i = 1, 2,\ \|f - \alpha G - \lambda w_\infty\|_{Y\cap W} \leqslant \tau \right\}. \tag{1.12}$$

The quantity $R'(\lambda, \tau)$ satisfies

$$\lim_{\lambda \to \frac{1}{2}} R'(\lambda, \tau) = \lim_{\tau \to \infty} R'(\lambda, \tau) = \infty.$$

Remark 1.1. It is not difficult to see that the constant $R(\lambda)$ can be taken as zero for sufficiently small λ. So the above theorems improve the existence result obtained by [6]. In [5] the solution of $(B_{\lambda,\alpha})$ is obtained in the polynomial weighted L^2 space for $\lambda \in [0, 1)$ when the Reynolds number $|\alpha|$ is sufficiently small. In particular, the solution is constructed near $\alpha \mathcal{G}_\lambda$ where

$$\mathcal{G}_\lambda(x) = \frac{\sqrt{1-\lambda^2}}{4\pi} e^{-\frac{1+\lambda}{4}x_1^2 - \frac{1-\lambda}{4}x_2^2}. \tag{1.13}$$

Note that $\mathcal{G}_\lambda$ is a solution of $(\mathcal{L} + \lambda\mathcal{M})\mathcal{G}_\lambda = 0$. The above result shows that the dynamics of the Burgers vortex depends on the Reynolds number and has simpler structures as $|\alpha|$ is increasing, which gives the rigorous explanation for the numerical observation for $\lambda \in [0, \frac{1}{2})$.

Remark 1.2. The asymptotic estimate of $\omega_{\alpha,\lambda}$ at large Reynolds numbers (1.10) is formally obtained by H.K. Moffatt, S. Kida, and K. Ohkitani [11], and it is rigorously proved in Th. Gallay and C.E. Wayne [6] by establishing the uniform estimates for the operator $(\mathcal{L} - \alpha\Lambda_G)^{-1}$ and using the smallness of λ ($\ll \frac{1}{2}$). In our proof we use the advantage at large Reynolds numbers instead of the smallness of λ. The core estimates are obtained in Lemma 3.1, from which we see how the radially or non-radially symmetric parts of the Burgers vortex are influenced by the value of the Reynolds numbers.

Remark 1.3. The restriction $\lambda < \frac{1}{2}$ seems to be essential if we try to find the Burgers vortex in the Gaussian weighted L^2 space X; see (1.10) and the estimates in Lemma 3.1. We also note that the function $\mathcal{G}_\lambda$ belongs to X if and only if $\lambda < \frac{1}{2}$. So if we consider the case $\lambda \in [\frac{1}{2}, 1)$, we have to deal with the problem in other function spaces which allow functions with slower spatial decays. However, in such spaces Λ_G is no longer skew-symmetric and this causes serious difficulties to control $\alpha\Lambda_G$ when α is not sufficiently small. After this work was completed, the author obtained a new idea which can be applied to the case $\lambda < 1$. But several technical arguments are required especially in the case near $\lambda = 1$, so in this paper we focus only on the case $\lambda < \frac{1}{2}$; See [13] for discussions in the case $\lambda \in [\frac{1}{2}, 1)$.

Since the asymmetric Burgers vortices are stationary solutions, their stability is an important problem. It is proved in [6] that they are locally stable when $\lambda \ll \frac{1}{2}$. In [5] the local stability with respect to three-dimensional perturbations is obtained when $\lambda < 1$ and $|\alpha| \leqslant \epsilon(\lambda) \ll 1$. Also in our case we can show their local stability if $|\alpha|$ is sufficiently large.

Let X_1 be the closed subspace of X defined by

$$X_1 = \left\{ f \in X \;\Big|\; \int_{\mathbb{R}^2} x_i f(x)\,dx = 0,\ i = 1, 2 \right\}. \tag{1.14}$$

Theorem 1.3 *(Stability). The asymmetric Burgers vortices constructed in Theorem* 1.1 *are locally stable in X_1 at least for $|\alpha| \gg R(\lambda)$.*

The precise meaning of the local stability is stated in Section 6. Unfortunately we do not know whether all asymmetric Burgers vortices constructed in Theorem 1.1 are locally stable in X_1 or not. Although in the above theorem the stability is considered in X_1, in fact, by considering a suitable shift as in [4], the local stability in X also follows as a consequence of Theorem 1.3.

In order to prove the main theorems, we expand $(B_{\lambda,\alpha})$ around $\alpha G + \lambda w_\infty$. Then we get the equation for $w = \omega - \alpha G - \lambda w_\infty$:

$$(\mathcal{L} - \alpha \Lambda_G + \lambda \mathcal{M}) w = B(w, w) + \lambda \Lambda_{w_\infty} w + \lambda f_\lambda, \tag{1.15}$$

where

$$B(f, h) = (K * f, \nabla) h, \tag{1.16}$$

$$\Lambda_h f = B(h, f) + B(f, h), \tag{1.17}$$

for $h, f \in Y$.

The function f_λ is defined as

$$f_\lambda = -\mathcal{L} w_\infty + \lambda\big(B(w_\infty, w_\infty) - \mathcal{M} w_\infty\big). \tag{1.18}$$

To derive the function f_λ from $(B_{\lambda,\alpha})$, we used the definition of w_∞, and the facts that $\mathcal{L}G = 0$, $B(f, g) = 0$ for $f, g \in Y \cap \mathbb{P}_S X$. By direct calculations, we can check that the functions w_∞, $B(w_\infty, w_\infty)$, and $\mathcal{M} w_\infty$ belongs to $\mathbb{P}_{S^\perp} X$; see Section 2. Hence the function f_λ also belongs to $\mathbb{P}_{S^\perp} X$.

In general, the integro-differential operator Λ_h is not a closed operator in X. To avoid this inconvenience, as in [12], we consider the closure of Λ_G instead of Λ_G itself. For simplicity we write

$$\Lambda = \overline{\Lambda_G}; \quad \text{the closure of } \Lambda_G \text{ in } X, \qquad \Lambda_1 = \Lambda_{w_\infty}. \tag{1.19}$$

We see that the equation for $w = \omega - \alpha G - \lambda w_\infty$ is also invariant in X_1. Thus we consider the equation

$$\begin{cases} (\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M}) w = B(w, w) + \lambda \Lambda_1 w + f_\lambda, \quad x \in \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} w(x)\,dx = \int_{\mathbb{R}^2} x_i w(x)\,dx = 0, \quad i = 1, 2. \end{cases} \tag{$\tilde{B}_{\lambda,\alpha}$}$$

The reason why we consider the equation in X_1 instead of X is that the kernel of Λ coincides with $\mathbb{P}_S X_1$ $(= \mathbb{P}_S X)$ in this space; see Section 2.

Let us state what is the difficulty and how we overcome it. The main difficulty appears when we deal with the term $\lambda \mathcal{M} w$. In [6] this term is treated as the perturbation. However, since $\mathcal{M} w$ is not a lower order term, we cannot regard the term $\lambda \mathcal{M} w$ as the perturbation if λ is not sufficiently small.

In [12] it is shown that the operator norm of the inverse of $\mathcal{L}_\alpha := (\mathcal{L} - \alpha \Lambda)|_{\mathbb{P}_{S^\perp} X}$ in X is small when $|\alpha|$ is large. But this is still not enough to control the term $\lambda \mathcal{M} w$. We note that if $\|\mathcal{L}_\alpha^{-1} \nabla\|_{X \to Y}$ is small when $|\alpha|$ is large, then we could regard $\lambda \mathcal{M} w$ as the perturbation. But it seems that the smallness of $\|\mathcal{L}_\alpha^{-1} \nabla\|_{X \to Y}$ is not true. So far, we only know $\|\mathcal{L}_\alpha^{-1} \nabla\|_{X \to Y}$ is uniformly bounded with respect to $|\alpha|$ by [6].

The above observation implies that we should treat the term $\lambda \mathcal{M} w$ as the main part of the equation when λ is not small. That is, we regard the term $(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M}) w$ as the principal term. Thus the most important step is to establish the estimates for the operator $(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1}$ in X_1. We note that even the existence of $(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1}$ is not trivial. The estimates for $\mathcal{L}_\alpha^{-1}$ suggest that $(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1}$ have better estimates if it acts on $\mathbb{P}_{S^\perp} X_1$ (the orthogonal complement of $\mathbb{P}_S X_1$ in X_1) and $|\alpha|$ is large. This is shown to be true, but we need several steps to prove this, since the term $\lambda \mathcal{M} w$ leads to the slow spatial decay in x_2 direction and also gives rise to the interaction between different Fourier modes with respect to the angular variable in polar coordinates. For example, we can easily see that the space $\mathbb{P}_S X_1$ or $\mathbb{P}_{S^\perp} X_1$ is not invariant under the action of the operator $\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M}$. With careful analyses of the interaction between the radially symmetric part and the non-radially symmetric part, we establish the required estimates for $(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1}$; see Section 3 for details.

We construct solutions of $(\tilde{B}_{\lambda,\alpha})$ based on the estimates for the operator $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$. To make use of the advantage at large Reynolds numbers, we decompose a solution of $(\tilde{B}_{\lambda,\alpha})$ into the radially symmetric part and non-radially symmetric part. For the non-radially symmetric part we obtain better estimates when $|\alpha|$ is large. On the other hand, we do not have any advantage in the estimates of $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$ for the radially symmetric part. However, from the structure of $(\tilde{B}_{\lambda,\alpha})$, we see that the radially symmetric part of solutions is essentially expressed by the non-radially symmetric part of them. This enables us to obtain the desired a priori estimates also for the radially symmetric part of solutions. The asymptotic estimate of solutions at large Reynolds numbers directly follows from the estimates of the function $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}f_\lambda$.

In order to prove the local stability of $\omega_{\alpha,\lambda}$, we investigate the spectral property of the linearization around $\omega_{\alpha,\lambda}$ and show that its spectral bound is away from zero if $|\alpha|$ is sufficiently large. Its proof is based on the contradiction argument. The main ingredients are the asymptotic expansion of $\omega_{\alpha,\lambda}$ in Theorem 1.1 and the spectral properties of Λ obtained in [12].

This paper is organized as follows. In Section 2 we summarize known results for some linear operators obtained in [4,6,12]. We also prove some properties of the bilinear form $B(f,h)$ and the function w_∞. In Section 3 we establish the estimates for the operator $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$, which is the core of this paper. In Section 4 we construct a solution of Eq. $(\tilde{B}_{\lambda,\alpha})$ which gives the proof of the former part of Theorem 1.1. Theorem 1.2 is also obtained in this section. In Section 5 we give the asymptotic estimates (1.10) by deriving the estimates of the function $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}f_\lambda$. In Section 6 we prove the local stability of the asymmetric Burgers vortices for sufficiently large $|\alpha|$.

2. Preliminaries

2.1. Known results for some linear operators

In this section we recall several known properties for some linear operators we consider in this paper.

First of all, it is well known that the operator $\mathcal{L}$ is self-adjoint in X and its spectrum consists of eigenvalues $\{-\frac{n}{2} \mid n=1,2,\dots\}$. The associated eigenfunctions for $-\frac{n}{2}$ are the Hermite functions $\{\partial_1^{\beta_1}\partial_2^{\beta_2}G\}$ with $\beta_1+\beta_2=n$. So the subspace X_1 is nothing but the orthogonal complement of $\{\beta_1\partial_1 G+\beta_2\partial_2 G \mid \beta_i\in\mathbb{R}\}$ in X.

In [6] and [5] Th. Gallay and C.E. Wayne proved the following lemma for the operators Λ_G and $\mathcal{L}-\alpha\Lambda_G$.

Lemma 2.1. *(See [5,6].)*

(1) $(-\mathcal{L})^{-\frac{1}{2}}$ *is bounded from X into $Y\cap W$.*
(2) Λ_G *is bounded from Y into X.*
(3) Λ_G *is skew-symmetric: for any $w_1, w_2\in Y$, we have $\langle \Lambda_G w_1, w_2\rangle_X+\langle w_1,\Lambda_G w_2\rangle_X=0$.*
(4) $(\mathcal{L}-\alpha\Lambda_G)^{-1}$ *is compact in X and bounded from X into Y. Moreover, its operator norm is bounded uniformly in α.*

In [12] the operator Λ (the closure of Λ_G in X) and $\mathcal{L}-\alpha\Lambda$ are studied. The operator Λ is expressed in terms of polar coordinates and the Fourier series expansion with respect to the angular variable; see [12, Section 2] for details. We only state the results in [12] without proofs. Let $\operatorname{Ker}\Lambda$ and $\operatorname{Ran}\Lambda$ be the kernel and the range of Λ, respectively.

Lemma 2.2. *(See [12].) The kernel of Λ in X is given by*

$$\operatorname{Ker}\Lambda=\mathbb{P}_S X\oplus\{\beta_1\partial_1 G+\beta_2\partial_2 G \mid \beta_i\in\mathbb{R}\}. \tag{2.1}$$

Moreover, let $\overline{\operatorname{Ran}\Lambda}$ be the closure of $\operatorname{Ran}\Lambda$ in X and let $\mathcal{L}_\alpha:=(\mathcal{L}-\alpha\Lambda)|_{\overline{\operatorname{Ran}\Lambda}}: D(\mathcal{L})\cap\overline{\operatorname{Ran}\Lambda}\to\overline{\operatorname{Ran}\Lambda}$. Then we have

$$\lim_{|\alpha|\to\infty}\sup_{\mu\in\sigma(\mathcal{L}_\alpha)}\operatorname{Re}(\mu)=-\infty. \tag{2.2}$$

Here, $\sigma(\mathcal{L}_\alpha)$ is the spectrum of $\mathcal{L}_\alpha$ and $\operatorname{Re}(\mu)$ is the real part of μ.

The above characterization of $\operatorname{Ker}\Lambda$ shows that $\operatorname{Ker}\Lambda=\mathbb{P}_S X_1$ if Λ is restricted on X_1. This fact is essentially used in this paper.

2.2. The properties of the bilinear form and the function w_∞

The bilinear form $B(f,h)=(K*f,\nabla)h$ plays important roles in the study of Burgers vortices. We start from the following proposition.

Proposition 2.1. *Let $2<r<3$ and $p=\frac{r}{2r-3}$. Let $f\in L^r(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ and $h\in Y$. Then we have*

$$\|B(f,h)\|_X\leqslant C\|f\|_{L^p}^{\frac{1}{4}}\|f\|_{L^r}^{\frac{3}{4}}\|h\|_Y, \tag{2.3}$$

$$\|(-\mathcal{L})^{-\frac{1}{2}}B(f,h)\|_X\leqslant C\|f\|_{L^p}^{\frac{1}{4}}\|f\|_{L^r}^{\frac{3}{4}}\|h\|_X. \tag{2.4}$$

Proof. We first note that by the Gagliardo–Nirenberg inequality, we have

$$\|K * f\|_{L^\infty} \leqslant C\|K * f\|_{L^q}^{\frac{1}{4}} \|\nabla K * f\|_{L^r}^{\frac{3}{4}},$$

where $\frac{1}{3q} = \frac{1}{2} - \frac{1}{r}$. We note that $2 < q < \infty$ from the condition $2 < r < 3$. Then by the Hardy–Littlewood–Sobolev inequality and the Calderón–Zygmund inequality, we see

$$\|K * f\|_{L^\infty} \leqslant C\|f\|_{L^p}^{\frac{1}{4}} \|f\|_{L^r}^{\frac{3}{4}}, \tag{2.5}$$

since $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$. Thus

$$\big\|B(f,h)\big\|_X = \big\|G^{-\frac{1}{2}}(K * f, \nabla)h\big\|_{L^2} \leqslant \|K * f\|_{L^\infty} \big\|G^{-\frac{1}{2}}\nabla h\big\|_{L^2} \leqslant C\|f\|_{L^p}^{\frac{1}{4}} \|f\|_{L^r}^{\frac{3}{4}} \|h\|_Y.$$

This proves the estimate (2.3).

To show the estimate (2.4), we prove the estimate

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\partial_i w\big\|_X \leqslant C\|w\|_X, \quad i = 1, 2, \tag{2.6}$$

for $w \in X$. This estimate is obtained by the duality argument. Indeed, we have for any $h \in X$,

$$\big\langle(-\mathcal{L})^{-\frac{1}{2}}\partial_i w, h\big\rangle_X = \big\langle\partial_i w, (-\mathcal{L})^{-\frac{1}{2}}h\big\rangle_X = -\frac{1}{2}\big\langle w, x_i(-\mathcal{L})^{-\frac{1}{2}}h\big\rangle_X - \big\langle w, \partial_i(-\mathcal{L})^{-\frac{1}{2}}h\big\rangle_X.$$

Since $(-\mathcal{L})^{-\frac{1}{2}}$ is bounded from X into $Y \cap W$, we have

$$\big|\big\langle(-\mathcal{L})^{-\frac{1}{2}}\partial_i w, h\big\rangle_X\big| \leqslant C\|w\|_X\|h\|_X,$$

this proves (2.6). Now the estimate (2.4) immediately follows from $B(f,h) = \nabla \cdot (hK * f)$ and the estimate (2.5). The proof of the proposition is completed. □

From the above proposition we can obtain the estimates for the integro-differential operator Λ_h. We set $A = (-\mathcal{L})^{\frac{1}{2}}$. Then A is sectorial; for example, see [3, Section II, Corollary 4.7]. Since A has a bounded inverse, we set the norm on $D(A^\gamma)$ for $\gamma \in [0,1]$ as

$$\|f\|_{D(A^\gamma)} = \big\|A^\gamma f\big\|_X, \tag{2.7}$$

instead of the usual graph norm. By the interpolation arguments, we have the following corollary.

Corollary 2.1. *Let $\gamma_1, \gamma_2 \in (0,1]$. Let $f \in D(A^{\gamma_1})$ and $h \in D(A^{\gamma_2})$. Then we have*

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Lambda_h f\big\|_X \leqslant C\big(\|h\|_{D(A^{\gamma_2})}\|f\|_X + \|f\|_{D(A^{\gamma_1})}\|h\|_X\big), \tag{2.8}$$

where C depends only on γ_1 and γ_2.

Proof. Let $2 < r < 3$. Then by the Gagliardo–Nirenberg inequality, we have

$$\|f\|_{L^r} \leqslant C\|f\|_{L^2}^{1-\sigma}\|\nabla f\|_{L^2}^{\sigma},$$

for $\sigma = 1 - \frac{2}{r}$. Thus $\|f\|_{L^r} \leqslant C\|f\|_X^{1-\sigma}\|Af\|_X^{\sigma}$ and this shows that

$$\|f\|_{L^r} \leqslant C\|f\|_{(X,D(A))_{\sigma,1}} \leqslant C\|f\|_{D(A^{\sigma'})}, \tag{2.9}$$

for $\sigma < \sigma'$; see [10, Section 2.2] for details. We note that if $2 < r < 3$, then $p = \frac{r}{2r-3} \in (1,2)$. Hence

$$\|f\|_{L^p} = \big\|G^{\frac{1}{2}}G^{-\frac{1}{2}}f\big\|_{L^p} \leqslant C\|f\|_X,$$

by the Hölder inequality. Combining these, by choosing suitable r in the estimate (2.4), we obtain the estimate (2.8). □

To see the qualitative properties of the bilinear form $B(f,h)$, we consider the representation of $B(f,h)$ in polar coordinates.

Let $n \in \mathbb{Z}$ and let $\mathbb{P}_n$ be the orthogonal projection defined by

$$\mathbb{P}_n w = w_n(r)e^{in\theta},$$

$$w_n(r) = \frac{1}{2\pi}\int_0^{2\pi} w(r\cos\theta, r\sin\theta)e^{-in\theta}\,d\theta.$$

We set

$$\mathbb{P}_n X = \{\mathbb{P}_n w \mid w \in X\}. \tag{2.10}$$

Then we have the following proposition for $B(f,h)$.

Proposition 2.2. *Let $f \in \mathbb{P}_n X$ and $h \in Y \cap \mathbb{P}_m X$. Then $B(f,h) \in \mathbb{P}_{n+m} X$.*

Proof. We recall the argument of [4, Lemma 4.4]. Let $f = f_n(r)e^{in\theta}$ and $h = h_m(r)e^{im\theta}$ in polar coordinates. We set $v_f = (v_f^{(1)}, v_f^{(2)}) = K * f$. We write $v_f^{(1)} = v_r \cos\theta - v_\theta \sin\theta$ and $v_f^{(2)} = v_r \sin\theta + v_\theta \cos\theta$ where $v_r = \bar{v}_r(r)e^{in\theta}$ and $v_\theta = \bar{v}_\theta e^{in\theta}$. Then from $\operatorname{div} v_f = 0$ and $\operatorname{rot} v_f = f$, we obtain the linear ordinary differential equations for $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$

$$\bar{v}_r' + \frac{\bar{v}_r}{r} + in\frac{\bar{v}_\theta}{r} = 0, \tag{2.11}$$

$$\bar{v}_\theta' + \frac{\bar{v}_\theta}{r} - in\frac{\bar{v}_r}{r} = f_n. \tag{2.12}$$

When $n \neq 0$, by eliminating $\bar{v}_\theta$, we obtain the equation for $\Omega_n = \frac{1}{2in} r\bar{v}_r$

$$-\frac{1}{r}\left(r\Omega_n'\right)' + \frac{n^2}{r^2}\Omega_n - \frac{1}{2}f_n = 0. \tag{2.13}$$

By the decay at infinity and the local integrability conditions, solution of the above equation is written by

$$\Omega_n(f_n)(r) = \frac{1}{4|n|}\left(\int_0^r \left(\frac{s}{r}\right)^{|n|} s f_n(s)\,ds + \int_r^\infty \left(\frac{r}{s}\right)^{|n|} s f_n(s)\,ds\right). \tag{2.14}$$

The function $\bar{v}_\theta$ is obtained by $\bar{v}_r$. From the uniqueness of the equation

$$\Delta v_f = \nabla^\perp f = (-\partial_2 f, \partial_1 f),$$

we see that v_f is indeed expressed by the above $\bar{v}_r$ and $\bar{v}_\theta$.

Now by using the relation $\partial_1 = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta$ and $\partial_2 = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta$, we obtain

$$B(f,h) = \left(\bar{v}_r h_m' + \frac{im}{r}\bar{v}_\theta h_m\right)e^{i(n+m)\theta}. \tag{2.15}$$

When $n = 0$, again by the decay at infinity and the local integrability conditions, we see that $\bar{v}_r = 0$ and $\bar{v}_\theta(r) = \frac{1}{r}\int_0^r s f_0(s)\,ds$ from (2.11), (2.12). Thus

$$B(f,h) = \frac{im}{r}\bar{v}_\theta h_m e^{im\theta}. \tag{2.16}$$

This completes the proof. □

Corollary 2.2. *If $h \in Y \cap \mathbb{P}_S X$, then $\Lambda_h f \in \mathbb{P}_{S^\perp} X$ for any $f \in Y \cap \mathbb{P}_{S^\perp} X$.*

Proof. Since $\mathbb{P}_S X = \mathbb{P}_0 X$, the assertion immediately follows from the above proposition. □

Corollary 2.3. *The function f_λ belongs to $\mathbb{P}_{S^\perp} X$.*

Proof. We recall that

$$f_\lambda = -\mathcal{L}w_\infty + \lambda\big(B(w_\infty, w_\infty) - \mathcal{M}w_\infty\big).$$

In [6, Proposition 3.1] the function w_∞ is obtained as $w_\infty = \overline{w}(r)\sin 2\theta$ for some function $\overline{w}(r)$. Note that from the characterization of $\operatorname{Ker}\Lambda$, this is uniquely determined in $\mathbb{P}_{S^\perp} X$.

Since $\mathbb{P}_{S^\perp} X$ is invariant under the action of $\mathcal{L}$, we have $\mathcal{L}w_\infty \in \mathbb{P}_{S^\perp} X$. By direct calculations, we also have $\mathcal{M}w_\infty \in \mathbb{P}_{S^\perp} X$. Moreover, from the above proposition and $w_\infty = \overline{w}(r)\sin 2\theta = \overline{w}(r)\frac{e^{i2\theta} - e^{-i2\theta}}{2i}$, it is not difficult to see $B(w_\infty, w_\infty) \in \mathbb{P}_{S^\perp} X$. Indeed, it suffices to show

$$B\big(\overline{w}e^{i2\theta}, \overline{w}e^{-i2\theta}\big) + B\big(\overline{w}e^{-i2\theta}, \overline{w}e^{i2\theta}\big) = 0, \tag{2.17}$$

by Proposition 2.2. From (2.14) we see $\Omega_2(\overline{w}) = \Omega_{-2}(\overline{w})$. Then from the representations (2.11), (2.12), and (2.15), we easily get (2.17). This completes the proof. □

Finally, we give the following simple proposition, which guarantees that the space X_1 is invariant under Eq. $(\tilde{B}_{\lambda,\alpha})$.

Proposition 2.3. *Let $f, h \in Y$. Then $\Lambda_h f \in X_1$.*

Proof. We set $v_f = (v_f^{(1)}, v_f^{(2)}) = K * f$ and $v_h = (v_h^{(1)}, v_h^{(2)}) = K * h$. The proof is given by the integration by parts. Indeed, by the definition of Λ_h, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} x_1 \Lambda_h f\,dx &= \int_{\mathbb{R}^2} x_1 \nabla\cdot(v_h f + v_f h)\,dx \\
&= -\int_{\mathbb{R}^2} \big(v_h^{(1)} f + v_f^{(1)} h\big)\,dx \\
&= \int_{\mathbb{R}^2} -v_h^{(1)}\big(-\partial_2 v_f^{(1)} + \partial_1 v_f^{(2)}\big) - v_f^{(1)}\big(-\partial_2 v_h^{(1)} + \partial_1 v_h^{(2)}\big)\,dx \\
&= \int_{\mathbb{R}^2} \big(-\partial_2 v_h^{(1)} v_f^{(1)} + \partial_1 v_h^{(1)} v_f^{(2)} + v_f^{(1)} \partial_2 v_h^{(1)} + \partial_1 v_f^{(1)} v_h^{(2)}\big)\,dx \\
&= -\int_{\mathbb{R}^2} \big(\partial_2 v_h^{(2)} v_f^{(2)} + \partial_2 v_f^{(2)} v_h^{(2)}\big)\,dx \\
&= 0.
\end{aligned}$$

Similarly, we have $\int_{\mathbb{R}^2} x_2 \Lambda_h f\,dx = 0$. It is obvious that $\int_{\mathbb{R}^2} \Lambda_h f\,dx = 0$. Now the proof is completed. □

3. The estimates for the linearized operator

In this section we establish the estimates for the linearized operator $\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M}$. The following lemma is the core of this paper. We recall that $A = (-\mathcal{L})^{\frac{1}{2}}$.

Lemma 3.1. *Let $\lambda \in [0, \frac{1}{2})$ and $\gamma \in [0, 1)$. Then there is some $R_1(\lambda) \geqslant 0$ independent of γ such that for any α with $|\alpha| \geqslant R_1(\lambda)$ and $f \in X_1$, we have*

$$\big\|(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} f\big\|_{Y\cap W} \leqslant \frac{K_1}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}} f\big\|_X, \tag{3.1}$$

$$\big\|\mathbb{P}_{S^\perp}(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} f\big\|_{D(A^\gamma)} \leqslant \delta_1\big(|\alpha|,\gamma\big)\bigg(\big\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} f\big\|_X + \frac{K_1\lambda}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}} f\big\|_X\bigg), \tag{3.2}$$

$$\begin{aligned}
\big\|\mathbb{P}_S(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} f\big\|_{D(A^\gamma)} &\leqslant (1+\lambda)\big\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}} f\big\|_X \\
&\quad + \lambda\delta_2\big(|\alpha|,\gamma\big)\bigg(\big\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} f\big\|_X + \frac{K_2\lambda}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}} f\big\|_X\bigg).
\end{aligned} \tag{3.3}$$

Here the constants K_1 and K_2 are independent of λ, γ, and α with $|\alpha| \geqslant R_1(\lambda)$. The constants $\delta_1(|\alpha|,\gamma)$ and $\delta_2(|\alpha|,\gamma)$ are bounded with respect to $|\alpha| \in [R_1(\lambda), \infty)$ and $\gamma \in [0, 1)$, and satisfy that

$$\lim_{|\alpha|\to\infty} \delta_1\big(|\alpha|,\gamma\big) = \lim_{|\alpha|\to\infty} \delta_2\big(|\alpha|,\gamma\big) = 0. \tag{3.4}$$

Remark 3.1. It is not difficult to see that the norm of $Y \cap W$ is equivalent with $\|\cdot\|_{D(A)}$. So the estimate (3.1) is corresponding to the case $\gamma = 1$ in (3.2) and (3.3), although in (3.1) we do not have better estimates for larger α like (3.4).

To prove the above lemma, we first consider the operator $\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I$. Since $\mathcal{L}$ is self-adjoint and $-\mathcal{L} \geqslant 1$ in X_1, we can write

$$\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I = -(-\mathcal{L})^{\frac{1}{2}}\big(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1}\big)(-\mathcal{L})^{\frac{1}{2}},$$

where

$$\Sigma = (-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{1}{2}}, \tag{3.5}$$

$$\Pi = (-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}. \tag{3.6}$$

By the results in [6, Lemmas 2.2, 2.3], we already know that Σ is compact and skew-symmetric in X and that Π is bounded in X. We shall show the following proposition.

Proposition 3.1. *Let $\lambda \in [0, \frac{1}{2})$. Then the operator $(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})$ has a bounded inverse in X satisfying the estimate*

$$\left\| \left(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1}\right)^{-1} f \right\|_X \leqslant \frac{1}{1-2\lambda} \|f\|_X. \tag{3.7}$$

Proof. Let $Q_{\alpha,\lambda}$ be a bilinear form on X defined by

$$Q_{\alpha,\lambda}(f,h) = \left\langle \left(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1}\right) f, h \right\rangle_X. \tag{3.8}$$

Clearly, $Q_{\alpha,\lambda}$ is bounded, i.e., there is some constant K such that $|Q_{\alpha,\lambda}(f,h)| \leqslant K\|f\|_X\|h\|_X$ for all $f, h \in X$. Since Σ is skew-symmetric, we have $\langle \Sigma f, f\rangle_X = 0$. We also recall the equality

$$\left\|(-\mathcal{L})^{\frac{1}{2}} h\right\|_X^2 = \left\langle h, (-\mathcal{L})h\right\rangle_X = \int_{\mathbb{R}^2} \left|\nabla\left(G^{-\frac{1}{2}}(x)h(x)\right)\right|^2 dx + \frac{1}{16}\left\||x|h\right\|_X^2 - \frac{1}{2}\|h\|_X^2,$$

which leads to the inequality

$$\left\|(-\mathcal{L})^{-\frac{1}{2}} f\right\|_X^2 \geqslant \frac{1}{8}\left\||x|(-\mathcal{L})^{-\frac{1}{2}} f\right\|_X^2 - 2\|f\|_X^2. \tag{3.9}$$

Combining these, we have

$$\begin{aligned}
Q_{\alpha,\lambda}(f,f) &= \left\langle \left(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1}\right) f, f \right\rangle_X \\
&\geqslant \|f\|_X^2 - \lambda\left\langle (-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}} f, f\right\rangle_X + \frac{\lambda}{8}\left\||x|(-\mathcal{L})^{-\frac{1}{2}} f\right\|_X^2 - 2\lambda\|f\|_X^2 \\
&= \|f\|_X^2 - \lambda\left\langle \mathcal{M}(-\mathcal{L})^{-\frac{1}{2}} f, (-\mathcal{L})^{-\frac{1}{2}} f\right\rangle_X + \frac{\lambda}{8}\left\||x|(-\mathcal{L})^{-\frac{1}{2}} f\right\|_X^2 - 2\lambda\|f\|_X^2 \\
&\geqslant \|f\|_X^2 - \frac{\lambda}{8}\left\||x|(-\mathcal{L})^{-\frac{1}{2}} f\right\|_X^2 + \frac{\lambda}{8}\left\||x|(-\mathcal{L})^{-\frac{1}{2}} f\right\|_X^2 - 2\lambda\|f\|_X^2 \\
&= (1-2\lambda)\|f\|_X^2,
\end{aligned}$$

thus $Q_{\alpha,\lambda}$ is coercive. By the Lax–Milgram theorem, $(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})$ is invertible in X, and the estimate (3.7) follows from the above inequality. This completes the proof of the proposition. □

From Proposition 3.1 we see that $\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I$ is invertible and its inverse $(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1} = -(-\mathcal{L})^{-\frac{1}{2}}(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})^{-1}(-\mathcal{L})^{-\frac{1}{2}}$ has the estimates

$$\left\|(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1} f\right\|_X \leqslant \frac{1}{1-2\lambda}\|f\|_X, \tag{3.10}$$

$$\left\|(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1} f\right\|_{Y\cap W} \leqslant \frac{C}{1-2\lambda}\|f\|_X. \tag{3.11}$$

Next we improve the above estimates for large $|\alpha|$. We set $h = (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1} f$. Then $h \in Y \cap W$ and we have

$$\left(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1}\right)(-\mathcal{L})^{\frac{1}{2}} h = -(-\mathcal{L})^{-\frac{1}{2}} f,$$

so

$$\left(I + \alpha\Sigma + \lambda(-\mathcal{L})^{-1}\right)(-\mathcal{L})^{\frac{1}{2}} h = -(-\mathcal{L})^{-\frac{1}{2}} f + \lambda\Pi(-\mathcal{L})^{\frac{1}{2}} h = -(-\mathcal{L})^{-\frac{1}{2}} f + \lambda(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h.$$

Thus we have the relation

$$h = -(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}(-\mathcal{L})^{-\frac{1}{2}} f + \lambda(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h,$$

where

$$\Gamma_{\alpha,\lambda} = I + \alpha\Sigma + \lambda(-\mathcal{L})^{-1}. \tag{3.12}$$

Let $\mathbb{P}_S$ be the projection from X_1 onto $\mathbb{P}_S X_1$; the closed subspace of all radially symmetric functions in X_1. Let $\mathbb{P}_{S^\perp} = I - \mathbb{P}_S$. We note that the projection $\mathbb{P}_S$ commutes with the operators $(-\mathcal{L})^{-\frac{1}{2}}$ and Σ. In fact, we have $\mathbb{P}_S\Sigma = 0$. Hence we can verify

$$\mathbb{P}_{S^\perp} h = -(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} f + \lambda(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h, \tag{3.13}$$

$$\mathbb{P}_S h = -(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}} f + \lambda(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}\mathbb{P}_{S^\perp} h. \tag{3.14}$$

In the last line we used the fact that

$$\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_S = \big(I+\alpha\Sigma+\lambda(-\mathcal{L})^{-1}\big)^{-1}\mathbb{P}_S = \big(I+\lambda(-\mathcal{L})^{-1}\big)^{-1}\mathbb{P}_S,$$

and $\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h = \mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}\mathbb{P}_{S^\perp}h$ for h with $\mathbb{P}_{S^\perp}h \in Y\cap W$. Note that $\mathbb{P}_{S^\perp}h\in Y\cap W$ follows from the representation (3.13), since $(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}$ is bounded from $Y\cap W$ to X.

The following lemma is crucial.

Lemma 3.2. *Let $\lambda \geqslant 0$ and $\gamma\in[0,1]$. Then we have for any $f\in X_1$,*

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S f\big\|_{D(A^\gamma)} \leqslant \|f\|_X, \tag{3.15}$$

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f\big\|_{D(A^\gamma)} \leqslant \epsilon_1\big(|\alpha|,\gamma\big)\|f\|_X, \tag{3.16}$$

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f\big\|_{D(A^\gamma)} \leqslant \epsilon_2\big(|\alpha|,\gamma\big)\|f\|_X. \tag{3.17}$$

Here the constants $\epsilon_1(|\alpha|,\gamma)$ and $\epsilon_2(|\alpha|,\gamma)$ are uniformly bounded with respect to $|\alpha|\geqslant 0$, $\gamma\in[0,1]$, and $\lambda\geqslant 0$. Moreover, when $\gamma\in[0,1)$ they satisfy that

$$\lim_{|\alpha|\to\infty}\epsilon_i\big(|\alpha|,\gamma\big)=0,\quad i=1,2. \tag{3.18}$$

Proof. First we note that

$$\big\|\Gamma_{\alpha,\lambda}^{-1}f\big\|_X \leqslant \|f\|_X. \tag{3.19}$$

Indeed, we have

$$\|\Gamma_{\alpha,\lambda}f\|_X \leqslant (1+\alpha+\lambda)C\|f\|_X,$$

and

$$\langle\Gamma_{\alpha,\lambda}f,f\rangle_X = \big\langle f+\alpha\Sigma f+\lambda(-\mathcal{L})^{-1}f,f\big\rangle_X = \|f\|_X^2+\lambda\big\|(-\mathcal{L})^{-\frac{1}{2}}f\big\|_X^2 \geqslant \|f\|_X^2.$$

These estimates give (3.19) by the Lax–Milgram theorem. From the estimate (3.19) we obtain

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S f\big\|_{D(A^\gamma)} \leqslant \|f\|_X,$$

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f\big\|_{D(A^\gamma)} \leqslant \|f\|_X,$$

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f\big\|_{D(A^\gamma)} \leqslant C'\|f\|_X,$$

where $C'=\|(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\|_{X\to X}$. Here we used the estimate $\|(-\mathcal{L})^{-\frac{1}{2}}f\|_{D(A^\gamma)}\leqslant\|f\|_X$ for $f\in X_1$.

We prove the estimates (3.16) and (3.17) by deriving a contradiction. Without loss of generality, we may assume that $\alpha>0$. Set $\epsilon_1(\alpha,\gamma) := \|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}\|_{X\to D(A^\gamma)}$.

We assume that $\limsup_{\alpha\to\infty}\epsilon_1(\alpha,\gamma)>0$. Then there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$, $\alpha_i\to\infty$ as $i\to\infty$, such that $\epsilon_1=\inf_{i\in\mathbb{N}}\epsilon_1(\alpha_i)>0$. So we have a sequence of functions $\{f_i\}_{i\in\mathbb{N}}$ with $\|f_i\|_X=1$ such that

$$\big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}f_i\big\|_{D(A^\gamma)} = \big\|(-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}f_i\big\|_X \geqslant \frac{\epsilon_1(\alpha_i)}{2}\|f_i\|_X \geqslant \frac{\epsilon_1}{2}>0.$$

We set $h_i=(-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}f_i\in\mathbb{P}_{S^\perp}X_1$.

Since $(-\mathcal{L})^{\frac{-1+\gamma}{2}}$ is compact (because $(\mathcal{L})^{-\frac{1}{2}}$ is compact) and $\{\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}f_i\}$ is bounded in X, we have a subsequence $\{h_j\}$ of $\{h_i\}$ such that h_j converges to a function $h_\infty\in\mathbb{P}_{S^\perp}X_1$ strongly in X_1. Then h_∞ satisfies $(-\mathcal{L})^{\frac{1-\gamma}{2}}h_\infty\in X_1$ and $\|h_\infty\|_X\geqslant\frac{\epsilon_1}{2}>0$.

On the other hand, for any $f\in X_1$, we see

$$\begin{aligned}
\big\langle(-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty,f\big\rangle_X &= -\big\langle(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty,\Lambda(-\mathcal{L})^{-\frac{1}{2}}f\big\rangle_X \\
&= -\lim_{j\to\infty}\big\langle(-\mathcal{L})^{-\frac{\gamma}{2}}h_j,\Lambda(-\mathcal{L})^{-\frac{1}{2}}f\big\rangle_X \\
&= \lim_{j\to\infty}\big\langle(-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_j,f\big\rangle_X \\
&= \lim_{j\to\infty}\frac{1}{\alpha_j}\Big(\big\langle\Gamma_{\alpha_j,\lambda}(-\mathcal{L})^{\frac{1-\gamma}{2}}h_j,f\big\rangle_X - \big\langle(-\mathcal{L})^{\frac{1-\gamma}{2}}h_j,f\big\rangle_X - \lambda\big\langle(-\mathcal{L})^{-\frac{1+\gamma}{2}}h_j,f\big\rangle_X\Big)
\end{aligned}$$

$$= \lim_{j\to\infty} \frac{1}{\alpha_j}\big(\langle \mathbb{P}_{S^\perp} f_j, f\rangle_X - \big\langle (-\mathcal{L})^{\frac{1-\gamma}{2}} h_j, f\big\rangle_X - \lambda\big\langle (-\mathcal{L})^{-\frac{1+\gamma}{2}} h_j, f\big\rangle_X\big)$$
$$= 0.$$

Thus $(-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty = 0$, that is, $\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty = 0$. However, since $\operatorname{Ker}\Lambda = \mathbb{P}_S X_1$ and $h_\infty \in \mathbb{P}_{S^\perp}X_1$ (and thus $(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty \in \mathbb{P}_{S^\perp}X_1$), we must have $(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty = 0$. Hence $h_\infty = 0$. This contradicts with $\|h_\infty\|_X > 0$. Now the estimate (3.16) has been proved.

From the estimate (3.16), we have the following claim:

Let $\{f_i\}_{i\in\mathbb{N}}$ be any bounded sequence in X_1. Then for any sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that $\alpha_i \to \infty$ as $i \to \infty$, the sequence $h_i = \Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i$ weakly converges to 0 in X_1.

Indeed, for any $f \in X_1 \cap D(\mathcal{L})$, we have

$$\lim_{i\to\infty}\langle h_i, f\rangle_X = \lim_{i\to\infty}\big\langle h_i, (-\mathcal{L})^{\frac{-1+\gamma}{2}}(-\mathcal{L})^{\frac{1-\gamma}{2}} f\big\rangle_X = \lim_{i\to\infty}\big\langle (-\mathcal{L})^{\frac{-1+\gamma}{2}} h_i, (-\mathcal{L})^{\frac{1-\gamma}{2}} f\big\rangle_X = 0.$$

Since $D(\mathcal{L})$ is dense in X_1 and h_i is bounded in X by the estimate (3.19), we have the claim.

The estimate (3.17) is shown by the above claim. We set

$$\epsilon_2(\alpha,\gamma) := \big\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}\big\|_{X\to D(A^\gamma)}$$
$$= \big\|(-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}\big\|_{X\to X}.$$

Again we assume that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$, $\alpha_i \to \infty$ as $i \to \infty$, satisfying $\epsilon_2 = \inf_{i\in\mathbb{N}}\alpha_i > 0$. Then we have $\{f_i\}_{i\in\mathbb{N}}$ with $\|f_i\|_X = 1$ such that

$$h_i = (-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i$$

satisfies $\inf_{i\in\mathbb{N}}\|h_i\|_X \geqslant \frac{\epsilon_2}{2} > 0$.

Since $(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}$ is bounded in X (note that $(-\mathcal{L})^{-\frac{1}{2}}x_i$ is bounded in X which is obtained in [6]), we have a subsequence $\{h_j\}_{j\in\mathbb{N}}$ of $\{h_i\}_{i\in\mathbb{N}}$ such that h_j strongly converges to a nontrivial function h_∞ in X_1. Now for any $f \in X_1$,

$$\langle h_\infty, f\rangle_X = \lim_{j\to\infty}\langle h_j, f\rangle_X$$
$$= \lim_{j\to\infty}\big\langle (-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp} f_j, f\big\rangle_X$$
$$= \lim_{j\to\infty}\big\langle \mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp} f_j, (-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}} f\big\rangle_X$$
$$= -\frac{1}{4}\lim_{j\to\infty}\big\langle (-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp} f_j, \big(x_1^2 - x_2^2\big)(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}} f\big\rangle_X$$
$$\quad - \lim_{j\to\infty}\big\langle (-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp} f_j, \mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}} f\big\rangle_X$$
$$= -\frac{1}{4}\lim_{j\to\infty}\big\langle \Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp} f_j, (-\mathcal{L})^{-\frac{1}{2}}\big(x_1^2 - x_2^2\big)(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}} f\big\rangle_X$$
$$\quad - \lim_{j\to\infty}\big\langle \Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp} f_j, (-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}} f\big\rangle_X$$
$$= 0,$$

by the above claim. This implies $h_\infty = 0$, which leads to a contradiction. Now the proof of the lemma is completed. □

Proof of Lemma 3.1. Let $\tilde{f} \in X_1$. We consider a solution h of the equation

$$(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})h = \tilde{f}.$$

Then, h satisfies the equation

$$(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)h = \tilde{f} - \lambda h. \tag{3.20}$$

Thus, from the estimate (3.11) for $f = \tilde{f} - \lambda h$, we have the estimate

$$\|h\|_{Y\cap W} \leqslant \frac{C}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}}(\tilde{f} - \lambda h)\big\|_X \leqslant \frac{C}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\big\|_X + \frac{C\lambda}{1-2\lambda}\big(\|\mathbb{P}_{S^\perp}h\|_X + \|\mathbb{P}_S h\|_X\big). \tag{3.21}$$

For simplicity, we write $\epsilon_1 = \epsilon_1(|\alpha|, \gamma), \epsilon_2 = \epsilon_2(|\alpha|, \gamma)$ in Lemma 3.2. We apply Lemma 3.2 to the expression (3.13). Then we have

$$\begin{aligned}\|\mathbb{P}_{S^\perp} h\|_{D(A^\gamma)} &\leqslant \epsilon_1\bigl(\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}h\bigr\|_X\bigr) + \lambda\epsilon_1\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h\bigr\|_X \\ &\leqslant \epsilon_1\bigl(\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda\|\mathbb{P}_{S^\perp}h\|_X\bigr) + C\lambda\epsilon_1\|h\|_Y,\end{aligned}$$

so

$$\|\mathbb{P}_{S^\perp} h\|_{D(A^\gamma)} \leqslant 2\epsilon_1\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + 2C\lambda\epsilon_1\|h\|_Y, \tag{3.22}$$

if $|\alpha|$ is sufficiently large or λ is sufficiently small.

We also have from (3.14) and Lemma 3.2,

$$\begin{aligned}\|\mathbb{P}_S h\|_{D(A^\gamma)} &\leqslant \bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h\bigr\|_{D(A^\gamma)} \\ &\quad + \lambda\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}(-\tilde{f} + \lambda h + \lambda\mathcal{M}h)\bigr\|_{D(A^\gamma)} \\ &\leqslant \bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h\bigr\|_{D(A^\gamma)} \\ &\quad + \lambda\epsilon_2\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda^2\epsilon_2\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}(h + \mathcal{M}h)\bigr\|_X \\ &\leqslant \bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h\bigr\|_{D(A^\gamma)} \\ &\quad + \lambda\epsilon_2\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + C\lambda^2\epsilon_2\|h\|_Y.\end{aligned}$$

By the estimate $\|(\mathcal{L} - \lambda I)^{-1} f\|_X \leqslant (1+\lambda)^{-1}\|f\|_X$ for any $f \in X_1$, we see that

$$\begin{aligned}\lambda\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h\bigr\|_{D(A^\gamma)} &= \lambda\bigl\|(-\mathcal{L})^{\frac{\gamma}{2}}(-\mathcal{L} + \lambda I)^{-1}\mathbb{P}_S h\bigr\|_X \\ &\leqslant \frac{\lambda}{1+\lambda}\bigl\|(-\mathcal{L})^{\frac{\gamma}{2}}\mathbb{P}_S h\bigr\|_X \\ &\leqslant \frac{\lambda}{1+\lambda}\|\mathbb{P}_S h\|_{D(A^\gamma)}.\end{aligned} \tag{3.23}$$

Hence we obtain

$$\|\mathbb{P}_S h\|_{D(A^\gamma)} \leqslant (1+\lambda)\bigl(\bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda\epsilon_2\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + C\lambda^2\epsilon_2\|h\|_Y\bigr). \tag{3.24}$$

Combining the estimates (3.22) and (3.24) for $\gamma = 0$, we have

$$\|\mathbb{P}_{S^\perp} h\|_X + \|\mathbb{P}_S h\|_X \leqslant (1+\lambda)\bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \bigl(2\epsilon_1 + \lambda(1+\lambda)\epsilon_2\bigr)\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \bigl(2\epsilon_1\lambda + C(1+\lambda)\lambda^2\epsilon_2\bigr)\|h\|_Y,$$

thus substituting this into (3.21), we get

$$\begin{aligned}\|h\|_{Y\cap W} &\leqslant \frac{2}{1-2\lambda}\bigl(C + (1+\lambda)\bigr)\bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \frac{2}{1-2\lambda}\bigl(C + 2\epsilon_1 + \lambda(1+\lambda)\epsilon_2\bigr)\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X \\ &\leqslant \frac{K_1}{1-2\lambda}\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X,\end{aligned} \tag{3.25}$$

if $|\alpha|$ is sufficiently large or λ is sufficiently small. This estimate proves the existence and boundedness of $(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}$ by the Fredholm alternative.

By substituting (3.25) into (3.22) and (3.24), we finally obtain

$$\|\mathbb{P}_{S^\perp} h\|_{D(A^\gamma)} \leqslant 2\epsilon_1\left(\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \frac{K_1\lambda}{1-2\lambda}\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X\right), \tag{3.26}$$

$$\|\mathbb{P}_S h\|_{D(A^\gamma)} \leqslant (1+\lambda)\bigl\|\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \lambda(1+\lambda)\epsilon_2\left(\bigl\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X + \frac{K_2\lambda}{1-2\lambda}\bigl\|(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}\bigr\|_X\right). \tag{3.27}$$

The proof of Lemma 3.1 is now completed. □

4. Construction of solutions

In this section we construct a solution of the equation

$$w = (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\big(B(w,w) + \lambda\Lambda_1 w + \lambda f_\lambda\big), \tag{4.1}$$

where Λ_1 and f_λ are given by (1.19) and (1.18). Functions considered in this section are always real-valued.

In order to use the estimates in Lemma 3.1 effectively, we decompose the equation into the radially symmetric part and the non-radially symmetric part. That is, we construct a solution of the form

$$w = w_S + w_{S^\perp}, \quad w_S \in Y \cap W \cap \mathbb{P}_S X_1, \ w_{S^\perp} \in Y \cap W \cap \mathbb{P}_{S^\perp} X_1.$$

Then we see

$$\begin{aligned} B(w,w) &= B(w_S, w_{S^\perp}) + B(w_{S^\perp}, w_S) + B(w_{S^\perp}, w_{S^\perp}) \\ &= \Lambda_{w_S} w_{S^\perp} + \frac{1}{2}\Lambda_{w_{S^\perp}} w_{S^\perp}, \\ \Lambda_1 w &= \Lambda_1 w_S + \Lambda_1 w_{S^\perp}. \end{aligned}$$

Note that the functions $\Lambda_{w_S} w_{S^\perp}$, $\Lambda_1 w_S$, and f_λ belong to $\mathbb{P}_{S^\perp} X_1$ by Corollary 2.2, Corollary 2.3 and Proposition 2.3.

We identify $D(A^\gamma)$ in X_1 with $\mathbb{P}_S D(A^\gamma) \times \mathbb{P}_{S^\perp} D(A^\gamma)$. Here $\mathbb{P}_S D(A^\gamma) = D(A^\gamma) \cap \mathbb{P}_S X_1$ and $\mathbb{P}_{S^\perp} D(A^\gamma) = D(A^\gamma) \cap \mathbb{P}_{S^\perp} X_1$.

For $(f,h) \in \mathbb{P}_S D(A^\gamma) \times \mathbb{P}_{S^\perp} D(A^\gamma)$, we set

$$H_1(f,h) = (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\Lambda_f h, \tag{4.2}$$

$$H_2(f,h) = \frac{1}{2}(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\Lambda_h h, \tag{4.3}$$

$$H_3(f,h) = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\Lambda_1 f, \tag{4.4}$$

$$H_4(f,h) = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\Lambda_1 h, \tag{4.5}$$

$$F_{\alpha,\lambda} = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} f_\lambda, \tag{4.6}$$

and

$$H_{\alpha,\lambda}(f,h) = \sum_{i=1}^{4} H_i(f,h) + F_{\alpha,\lambda}. \tag{4.7}$$

The term $(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\Lambda_f h$ makes sense for any $f, h \in D(A^\gamma)$ with $\gamma \in (0,1]$. Indeed, by Lemma 3.1 and (2.8), we have

$$\big\|(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}\Lambda_f h\big\|_{Y\cap W} \leqslant \frac{K_1}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}}\Lambda_f h\big\|_X \leqslant \frac{CK_1}{1-2\lambda}\big(\|h\|_{D(A^\gamma)}\|f\|_X + \|f\|_{D(A^\gamma)}\|h\|_X\big).$$

Thus the above $H_{\alpha,\lambda}$ maps $\mathbb{P}_S D(A^\gamma) \times \mathbb{P}_{S^\perp} D(A^\gamma)$ into $Y \cap W \cap X_1$ for any $\gamma \in (0,1]$.

We fix $\gamma \in (0,1)$ and write $D_S = \mathbb{P}_S D(A^\gamma)$, $D_{S^\perp} = \mathbb{P}_{S^\perp} D(A^\gamma)$ for simplicity.

Now we define the map $\Phi_{\alpha,\lambda}$ on $D_S \times D_{S^\perp}$ by

$$\Phi_{\alpha,\lambda}(f,h) = \big(\mathbb{P}_S H_{\alpha,\lambda}(f,h), \mathbb{P}_{S^\perp} H_{\alpha,\lambda}(f,h)\big). \tag{4.8}$$

By Lemma 3.1, this map $\Phi_{\alpha,\lambda}$ is well defined. Let $\kappa_1, \kappa_2 > 0$ and let X_{κ_1,κ_2} be a closed convex subset in $D_S \times D_{S^\perp}$ such that

$$X_{\kappa_1,\kappa_2} = \big\{(f,h) \in D_S \times D_{S^\perp} \bigm| \|f\|_{D(A^\gamma)} \leqslant \kappa_1, \|h\|_{D(A^\gamma)} \leqslant \kappa_2\big\}. \tag{4.9}$$

The following proposition leads to Theorem 1.1.

Proposition 4.1. *Let $\lambda \in [0, \frac{1}{2})$. Then there exist $\kappa_1(\lambda)$, $\kappa_2(\lambda)$, and $R_2(\lambda) \geqslant 0$ such that for any α with $|\alpha| \geqslant R_2(\lambda)$, the above $\Phi_{\alpha,\lambda}$ has a fixed point in $X_{\kappa_1(\lambda),\kappa_2(\lambda)}$.*

Proof. First we show that $\Phi_{\alpha,\lambda}$ is a completely continuous mapping on $X_{\kappa_1(\lambda),\kappa_2(\lambda)}$ into itself for suitable $\kappa_1(\lambda)$, $\kappa_2(\lambda)$, and $R_1(\lambda) \geqslant 0$. By Lemma 3.1 we have

$$\begin{aligned} \big\|\mathbb{P}_S H_1(f,h)\big\|_{D(A^\gamma)} &\leqslant \lambda\delta_2\left(\big\|\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\Lambda_f h\big\|_X + \frac{K_2\lambda}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac{1}{2}}\Lambda_f h\big\|_X\right) \\ &\leqslant C\lambda\left(1 + \frac{\lambda}{1-2\lambda}\right)\delta_2\big(\|f\|_X\|h\|_{D(A^\gamma)} + \|f\|_{D(A^\gamma)}\|h\|_X\big). \end{aligned}$$

Here we used the estimates (2.8). Similarly we obtain from Lemma 3.1,

$$\left\|\mathbb{P}_{S^\perp} H_1(f,h)\right\|_{D(A^\gamma)} \leqslant C\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\left(\|f\|_X\|h\|_{D(A^\gamma)}+\|f\|_{D(A^\gamma)}\|h\|_X\right),$$

$$\left\|\mathbb{P}_S H_2(f,h)\right\|_{D(A^\gamma)} \leqslant C\left(1+\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_2\right)\|h\|_X\|h\|_{D(A^\gamma)},$$

$$\left\|\mathbb{P}_{S^\perp} H_2(f,h)\right\|_{D(A^\gamma)} \leqslant C\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\|h\|_X\|h\|_{D(A^\gamma)},$$

$$\left\|\mathbb{P}_S H_3(f,h)\right\|_{D(A^\gamma)} \leqslant C\lambda^2\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_2\|f\|_{D(A^\gamma)},$$

$$\left\|\mathbb{P}_{S^\perp} H_3(f,h)\right\|_{D(A^\gamma)} \leqslant C\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\|f\|_{D(A^\gamma)},$$

$$\left\|\mathbb{P}_S H_4(f,h)\right\|_{D(A^\gamma)} \leqslant C\lambda\left(1+\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_2\right)\|h\|_{D(A^\gamma)},$$

$$\left\|\mathbb{P}_{S^\perp} H_4(f,h)\right\|_{D(A^\gamma)} \leqslant C\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\|h\|_{D(A^\gamma)}.$$

In the estimates for H_3 we used the fact that $\Lambda_1 f \in \mathbb{P}_{S^\perp}X_1$ by Corollary 2.2. We also remark that the estimates for $\|\mathbb{P}_S H_2(f,h)\|_{D(A^\gamma)}$ and $\|\mathbb{P}_S H_4(f,h)\|_{D(A^\gamma)}$ imply that we potentially require the smallness of $\|h\|_{D(A^\gamma)}$ itself. Especially, the fact that the term $\mathbb{P}_S H_4(f,h)$ does not depend on f is crucial, since the prefactor constant is not sufficiently small when λ is not small enough.

The estimate for $F_{\alpha,\lambda}$ is

$$\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)} \leqslant C\lambda^2\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_2\left\|(-\mathcal{L})^{-\frac{1}{2}}f_\lambda\right\|_X,$$

$$\|\mathbb{P}_{S^\perp} F_{\alpha,\lambda}\|_{D(A^\gamma)} \leqslant C\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\left\|(-\mathcal{L})^{-\frac{1}{2}}f_\lambda\right\|_X,$$

hence especially we have

$$\lim_{|\alpha|\to\infty}\|F_{\alpha,\lambda}\|_{D(A^\gamma)}=0. \tag{4.10}$$

In fact, we have more precise estimates for $\|F_{\alpha,\lambda}\|_{D(A)}$; see Proposition 5.1. We also note that $\|F_{\alpha,\lambda}\|_{D(A^\gamma)}$ is sufficiently small uniformly in $|\alpha|$ if λ is sufficiently small. Combining these above, we obtain

$$\begin{aligned}\left\|\mathbb{P}_S H(f,h)\right\|_{D(A^\gamma)} \leqslant{}& C_0\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_2\big\{\|f\|_X\|h\|_{D(A^\gamma)}+\|f\|_{D(A^\gamma)}\|h\|_X\\&+\|h\|_X\|h\|_{D(A^\gamma)}+\lambda\left(\|f\|_{D(A^\gamma)}+\|h\|_{D(A^\gamma)}\right)\big\}\\&+C_0\left(\|h\|_X\|h\|_{D(A^\gamma)}+\lambda\|h\|_{D(A^\gamma)}\right)+\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\left\|\mathbb{P}_{S^\perp} H(f,h)\right\|_{D(A^\gamma)} \leqslant{}& C_0\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\big\{\|f\|_X\|h\|_{DA^\gamma)}+\|f\|_{D(A^\gamma)}\|h\|_X\\&+\|h\|_X\|h\|_{D(A^\gamma)}+\lambda\left(\|f\|_{D(A^\gamma)}+\|h\|_{D(A^\gamma)}\right)\big\}+\|\mathbb{P}_{S^\perp} F_{\alpha,\lambda}\|_{D(A^\gamma)}.\end{aligned} \tag{4.12}$$

Here C_0 is a numerical constant.

Let $\kappa_2 \leqslant \kappa_1 \leqslant 1$. We take $|\alpha|$ large (or λ small) enough to satisfy

$$C_0\lambda(1+\lambda)\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_2 \leqslant \frac{1}{N}, \tag{4.13}$$

where $N \gg 1$ is determined later. Then for $(f,h)\in X_{\kappa_1,\kappa_2}$, we have

$$\left\|\mathbb{P}_S H(f,h)\right\|_{D(A^\gamma)} \leqslant \frac{1}{4}\kappa_1 + C_0(1+\lambda)\left(\kappa_2^2+\lambda\kappa_2\right)+\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)}.$$

Next we consider the estimate of (4.12). If λ is not small, then we take $|\alpha|$ large enough to satisfy

$$C_0\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1 \leqslant \frac{1}{N}. \tag{4.14}$$

If λ is small enough to satisfy

$$C_0\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\sup_{|\alpha|}\delta_1\big(|\alpha|,\lambda\big)\leqslant\frac{1}{N},\tag{4.15}$$

then we take κ_1 and κ_2 sufficiently small such as

$$2C_0\left(1+\frac{\lambda}{1-2\lambda}\right)(\kappa_1+\kappa_2)\sup_{|\alpha|}\delta_1\big(|\alpha|,\lambda\big)\leqslant\frac{1}{N}.\tag{4.16}$$

In each case we have

$$\big\|\mathbb{P}_{S^\perp}H(f,h)\big\|_{D(A^\gamma)}\leqslant\frac{1}{4}\kappa_2+\frac{1}{N}(\kappa_1+\kappa_2)+\|\mathbb{P}_{S^\perp}F_{\alpha,\lambda}\|_{D(A^\gamma)}.$$

We take $8C_0(1+\lambda)\kappa_2=\kappa_1$. Then we have

$$\big\|\mathbb{P}_S H(f,h)\big\|_{D(A^\gamma)}\leqslant\frac{1}{2}\kappa_1+\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)},\tag{4.17}$$

$$\big\|\mathbb{P}_{S^\perp}H(f,h)\big\|_{D(A^\gamma)}\leqslant\frac{1}{4}\kappa_2+\frac{8C_0(1+\lambda)+1}{N}\kappa_2+\|\mathbb{P}_{S^\perp}F_{\alpha,\lambda}\|_{D(A^\gamma)}.\tag{4.18}$$

Thus if we take N such as $\frac{8C_0(1+\lambda)+1}{N}\leqslant\frac{1}{4}$, then $\Phi_{\alpha,\lambda}$ maps X_{κ_1,κ_2} into itself, because $\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)}$ and $\|\mathbb{P}_{S^\perp}F_{\alpha,\lambda}\|_{D(A^\gamma)}$ are sufficiently small if we take $|\alpha|$ large or λ small enough. We omit the details. Since H is a mapping from $D_S\times D_{S^\perp}$ into $D(A)$, it is easy to see that $\Phi_{\alpha,\lambda}$ is completely continuous.

Hence by the Schauder fixed point theorem, $\Phi_{\alpha,\lambda}$ has at least one fixed point on X_{κ_1,κ_2} (if we take α even larger, then we can apply the Banach fixed point theorem). The proof of Proposition 4.1 is completed. □

Proof of Theorem 1.2. Let $\tau_1,\tau_2>0$ and let $(f_1,h_1),(f_2,h_2)\in X_{\tau_1,\tau_2}$ be fixed points of $\Phi_{\alpha,\lambda}$. Then, by arguing in the same way as above, it is not difficult to see that if $|\alpha|$ is sufficiently large (depending on τ_1,τ_2), then we have

$$\|f_1-f_2\|_{D(A^\gamma)}=\big\|\mathbb{P}_S H(f_1,h_1)-\mathbb{P}_S H(f_2,h_2)\big\|_{D(A^\gamma)}\leqslant\frac{1}{4}\|f_1-f_2\|_{D(A^\gamma)}+C\|h_1-h_2\|_{D(A^\gamma)},$$

$$\|h_1-h_2\|_{D(A^\gamma)}=\big\|\mathbb{P}_{S^\perp}H(f_1,h_1)-\mathbb{P}_{S^\perp}H(f_2,h_2)\big\|_{D(A^\gamma)}\leqslant\frac{1}{4}\|h_1-h_2\|_{D(A^\gamma)}+\frac{C}{N}\|f_1-f_2\|_{D(A^\gamma)},$$

with $N\gg1$ and a constant C which depends only on τ_1 and τ_2.

Hence $\|f_1-f_2\|_{D(A^\gamma)}\leqslant\frac{4C}{3}\|h_1-h_2\|_{D(A^\gamma)}$ and

$$\|h_1-h_2\|_{D(A^\gamma)}\leqslant\frac{1}{4}\|h_1-h_2\|_{D(A^\gamma)}+\frac{4C^2}{3N}\|h_1-h_2\|_{D(A^\gamma)}\leqslant\frac{1}{2}\|h_1-h_2\|_{D(A^\gamma)},$$

if N is large enough. Thus we have $h_1=h_2$, and also $f_1=f_2$. This completes the proof of Theorem 1.2. □

5. Large Reynolds number asymptotics

In this section we prove the asymptotic estimate (1.10). Let $w_{\alpha,\lambda}$ be the solution obtained in Proposition 4.1. Let $0<\gamma<1$. Then it is not difficult to see

$$\|w_{\alpha,\lambda}\|_{D(A^\gamma)}\leqslant C\|F_{\alpha,\lambda}\|_{D(A^\gamma)},\tag{5.1}$$

for a numerical constant $C>0$ by the estimates (4.11) and (4.12). We give a proof only in the case of large $|\alpha|$ here. In this case we may assume that the constant $C_0(1+\frac{\lambda}{1-2\lambda})(\delta_1+\delta_2)$ in (4.11) and (4.12) is sufficiently small. Note that we already have

$$\|\mathbb{P}_S w_{\alpha,\lambda}\|_{D(A^\gamma)},\ \|\mathbb{P}_{S^\perp}w_{\alpha,\lambda}\|_{D(A^\gamma)}\leqslant1$$

by the proof of Proposition 4.1. Thus we obtain

$$\|\mathbb{P}_S w_{\alpha,\lambda}\|_{D(A^\gamma)}\leqslant C\|\mathbb{P}_{S^\perp}w_{\alpha,\lambda}\|_{D(A^\gamma)}+2\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)},$$

$$\|\mathbb{P}_{S^\perp}w_{\alpha,\lambda}\|_{D(A^\gamma)}\leqslant C\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\|\mathbb{P}_S w_{\alpha,\lambda}\|_{D(A^\gamma)}+2\|\mathbb{P}_{S^\perp}F_{\alpha,\lambda}\|_{D(A^\gamma)}.$$

Hence

$$\|\mathbb{P}_{S^\perp}w_{\alpha,\lambda}\|_{D(A^\gamma)}\leqslant C\lambda\left(1+\frac{\lambda}{1-2\lambda}\right)\delta_1\big(\|\mathbb{P}_{S^\perp}w_{\alpha,\lambda}\|_{D(A^\gamma)}+2\|\mathbb{P}_S F_{\alpha,\lambda}\|_{D(A^\gamma)}\big)+\|\mathbb{P}_{S^\perp}F_{\alpha,\lambda}\|_{D(A^\gamma)},$$

that is,

$$\|\mathbb{P}_{S^\perp} w_{\alpha,\lambda}\|_{D(A^\gamma)} \leqslant C\|F_{\alpha,\lambda}\|_{D(A^\gamma)}.$$

The estimate (5.1) is now easily obtained.

Since $w_{\alpha,\lambda}$ is a solution of Eq. (4.1), by the estimate (3.1), we have the estimate of $\|w_{\alpha,\lambda}\|_{Y\cap W}$ such that

$$\begin{aligned}\|w_{\alpha,\lambda}\|_{Y\cap W} &\leqslant \frac{C}{1-2\lambda}\big(\|w_{\alpha,\lambda}\|_{D(A^\gamma)}+\lambda\big)\|w_{\alpha,\lambda}\|_{D(A^\gamma)}+\|F_{\alpha,\lambda}\|_{Y\cap W}\\ &\leqslant \frac{C}{1-2\lambda}\big(\|F_{\alpha,\lambda}\|_{D(A^\gamma)}+\lambda\big)\|F_{\alpha,\lambda}\|_{D(A^\gamma)}+\|F_{\alpha,\lambda}\|_{Y\cap W}.\end{aligned} \tag{5.2}$$

Hence the large Reynolds number asymptotics of solutions is controlled by the behavior of $F_{\alpha,\lambda}=\lambda(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}f_\lambda$. By the arguments in [6] we obtain the desired estimate as follows.

Proposition 5.1. *Let $\lambda\in[0,\frac12)$ and let $R_2(\lambda)$ be the number obtained in Proposition* 4.1. *Then for any α with $|\alpha|\geqslant R_2(\lambda)$, the function $F_{\alpha,\lambda}$ satisfies*

$$\|F_{\alpha,\lambda}\|_{Y\cap W}\leqslant \frac{C\lambda}{(1-2\lambda)(1+|\alpha|)}. \tag{5.3}$$

Proof. We only give the proof for the case $|\alpha|\gg 1$. By Corollary 2.3 we already know $f_\lambda\in\mathbb{P}_{S^\perp}X_1=\overline{\operatorname{Ran}\Lambda}$. Moreover, by investigating the equation $\mathcal{M}G=\Lambda w_\infty$, we see that $f_\lambda\in\operatorname{Ran}\Lambda$ and the function h_λ satisfying $f_\lambda=\Lambda h_\lambda$ also belongs to $D(\mathcal{L})$; see [6, Section 3]. We omit the details here. Now we use the argument in [6, Proposition 3.4].

$$\begin{aligned}-(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}f_\lambda &= -(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}\Lambda h_\lambda\\ &= \frac{1}{\alpha}(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})h_\lambda-\frac{1}{\alpha}(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}(\mathcal{L}+\lambda\mathcal{M})h_\lambda\\ &= \frac{1}{\alpha}\big\{h_\lambda-(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}(\mathcal{L}+\lambda\mathcal{M})h_\lambda\big\}.\end{aligned}$$

Thus we have from (3.1),

$$\|F_{\alpha,\lambda}\|_{Y\cap W}\leqslant \frac{C\lambda}{|\alpha|}\left\{\|h_\lambda\|_{Y\cap W}+\frac{1}{1-2\lambda}\big\|(-\mathcal{L})^{-\frac12}(\mathcal{L}+\lambda\mathcal{M})h_\lambda\big\|_X\right\}\leqslant \frac{C\lambda}{(1-2\lambda)|\alpha|}\|h_\lambda\|_{Y\cap W}.$$

This gives the desired estimate for $|\alpha|\gg 1$. □

6. Stability of the Burgers vortices

In this section we study the stability of the Burgers vortices constructed in the previous section. Let $\omega_{\alpha,\lambda}$ be the solution to $(B_{\lambda,\alpha})$ in Theorem 1.1. Especially, $\omega_{\alpha,\lambda}$ satisfies the estimate

$$\|\omega_{\alpha,\lambda}-\alpha G-\lambda\omega_\infty\|_{Y\cap W}\leqslant \frac{C\lambda}{(1-2\lambda)(1+|\alpha|)}. \tag{6.1}$$

We expand the evolution equations associated with $(B_{\lambda,\alpha})$ around $\omega_{\alpha,\lambda}$, that is, we consider

$$\begin{cases}\partial_t w-(\mathcal{L}+\lambda\mathcal{M}-\Lambda_{\omega_{\alpha,\lambda}})w=-(K*w,\nabla)w, & t>0,\ x\in\mathbb{R}^2,\\ w(0,x)=w_0(x), & x\in\mathbb{R}^2,\end{cases} \tag{6.2}$$

where

$$\Lambda_{\omega_{\alpha,\lambda}}w=(K*\omega_{\alpha,\lambda},\nabla)w+(K*w,\nabla)\omega_{\alpha,\lambda}. \tag{6.3}$$

The local stability of $\omega_{\alpha,\lambda}$ in X_1 is stated as follows: there is an $\epsilon>0$ such that if $w_0\in X_1$ and $\|w_0\|_X\leqslant\epsilon$, then the solution $w(t)$ to (6.2) satisfies $\lim_{t\to\infty}\|w(t)\|_X=0$. As stated in the introduction, the stability in X follows from the stability in X_1 by considering a shift as in [4]. Indeed, when the initial data ω_0 has a nonzero first moment, if we set $\tilde\omega(t,x)=\omega(t,x_1+\frac{\beta_1}{\alpha}e^{-\frac{1+\lambda}{2}t},x_2+\frac{\beta_2}{\alpha}e^{-\frac{1-\lambda}{2}t})$ where $\beta_i=\int_{\mathbb{R}^2}x_i\omega_0(x)\,dx$, then $\tilde\omega(t)$ also satisfies the evolution equation associated with $(B_{\lambda,\alpha})$ for the initial data $\tilde\omega_0$ with zero first moments. Hence we can apply the above result for $\tilde w(t):=\tilde\omega(t)-\omega_{\alpha,\lambda}\in X_1$, which gives the stability in X together with the second order asymptotics by using a Taylor expansion. We omit the details here.

To prove the local stability of $\omega_{\alpha,\lambda}$ we study the spectrum of the linearized operator

$$L_{\alpha,\lambda}=\mathcal{L}+\lambda\mathcal{M}-\Lambda_{\omega_{\alpha,\lambda}}$$

in X_1. The following lemma is essential.

Lemma 6.1. *Let $\lambda \in [0, \frac{1}{2})$ and let $\sigma(L_{\alpha,\lambda})$ be the spectrum of $L_{\alpha,\lambda}$ in X_1. Then*

$$\limsup_{|\alpha|\to\infty} \sup_{\mu\in\sigma(L_{\alpha,\lambda})} \mathrm{Re}(\mu) \leqslant -1. \tag{6.4}$$

Proof. From Lemma 3.1 and (6.1) it is not difficult to see that $L_{\alpha,\lambda}$ has a compact resolvent. Thus it suffices to consider the eigenvalue problem in X_1

$$L_{\alpha,\lambda} w = \mu w. \tag{6.5}$$

The proof is based on a contradiction argument used in [12]. The key tools are the estimate (6.1) and the fact that Λ has no eigenvalues except for the eigenvalue zero; see [12, Lemma 5.1].

Assume that the assertion in Lemma 6.1 is not true. Then we have sequences $\{\mu_n\} \subset \mathbb{C}$ and $\{\alpha_n\} \subset \mathbb{R}$ such that $\inf_n \mathrm{Re}(\mu_n) > -1$, $|\alpha_n| \to \infty$ and

$$L_{\alpha_n,\lambda} w_n = \mu_n w_n,$$

where $w_n \in X_1 \cap D(\mathcal{L})$ and $\|w_n\|_X = 1$. Thus

$$\mu_n = \mu_n \|w_n\|_X^2 = \langle L_{\alpha_n,\lambda} w_n, w_n\rangle_X = \int_{\mathbb{R}^2} G^{-1}(\mathcal{L} + \lambda\mathcal{M} - \alpha_n \Lambda - \lambda \Lambda_{w_\infty} - \Lambda_{w_{\alpha_n,\lambda}}) w_n \overline{w_n}\, dx.$$

Here $w_{\alpha_n,\lambda} = \omega_{\alpha_n,\lambda} - \alpha_n G - \lambda w_\infty$. Let $f_n = G^{-\frac{1}{2}} w_n$. Taking the real part of both sides, we have from the skew-symmetry of Λ,

$$\begin{aligned}
\mathrm{Re}(\mu_n) &= -\int_{\mathbb{R}^2} \left(|\nabla f_n|^2 + \frac{|x|^2}{16}|f_n|^2 - \frac{1}{2}|f_n|^2 + \frac{\lambda}{8}\left(x_1^2 - x_2^2\right)|f_n|^2 \right) dx - \mathrm{Re}\int_{\mathbb{R}^2} G^{-1}(\lambda \Lambda_{w_\infty} w_n + \Lambda_{w_{\alpha_n,\lambda}} w_n)\overline{w_n}\, dx \\
&\leqslant -\int_{\mathbb{R}^2} \left(|\nabla f_n|^2 + \frac{(1-2\lambda)|x|^2}{16}|f_n|^2 - \frac{1}{2}|f_n|^2 \right) dx - \mathrm{Re}\int_{\mathbb{R}^2} G^{-1}(\lambda \Lambda_{w_\infty} w_n + \Lambda_{w_{\alpha_n,\lambda}} w_n)\overline{w_n}\, dx.
\end{aligned}$$

For $f, w \in Y \cap W$ we have $\|\Lambda_f w\|_X \leqslant C\|f\|_Y \|w\|_Y$ which can be obtained by Proposition 2.1. Hence by the Schwarz inequality we have

$$\left| \int_{\mathbb{R}^2} G^{-1}(\lambda \Lambda_{w_\infty} w_n + \Lambda_{w_{\alpha_n,\lambda}} w_n)\overline{w_n}\, dx \right| \leqslant \left(\|\Lambda_{w_\infty} w_n\|_X + \|\Lambda_{w_{\alpha_n,\lambda}} w_n\|_X \right)\|w_n\|_X \leqslant C\left(\|w_\infty\|_Y + \|w_{\alpha_n,\lambda}\|_Y \right)\|w_n\|_Y \|w_n\|_X.$$

Thus from $\|w_{\alpha,\lambda}\|_{Y\cap W} \leqslant \frac{C}{(1-2\lambda)(1+|\alpha|)}$ we easily see that

$$\mathrm{Re}(\mu_n) \leqslant C \int_{\mathbb{R}^2} |f_n|^2\, dx = C, \tag{6.6}$$

where C is independent of n. At the same time, we obtain the estimate

$$\|\nabla f_n\|_{L^2}^2 + \big\| |x| f_n \big\|_{L^2}^2 + \|f_n\|_{L^2}^2 \leqslant C \tag{6.7}$$

where $f_n = G^{-\frac{1}{2}} w_n$ and C is independent of n. Hence the sequence $\{w_n\}$ is compact in X and there is a subsequence $\{w_k\}$ which strongly converges to w^* in X as k goes to ∞. Moreover, w^* belongs to $X_1 \cap Y \cap W$ and $\|w^*\|_X = 1$. Now we claim that $\lim_{k\to\infty} \frac{\mathrm{Im}(\mu_k)}{\alpha_k}$ exists. Here, $\mathrm{Im}(\mu_k)$ is the imaginary part of μ_k. Indeed, from the equality $\langle L_{\alpha_k,\lambda} w_k, w_k\rangle_X = \mu_k$, taking the imaginary part of both sides, we get

$$\frac{\mathrm{Im}(\mu_k)}{\alpha_k} = \frac{1}{\alpha_k}\mathrm{Im}\left(\left\langle (\mathcal{L} + \lambda\mathcal{M}) w_k, w_k \right\rangle_X\right) - \mathrm{Im}\left(\langle \Lambda w_k, w_k\rangle_X\right) - \frac{1}{\alpha_k}\mathrm{Im}\left(\left\langle (\lambda\Lambda_{w_\infty} + \Lambda_{w_{\alpha_k,\lambda}}) w_k, w_k \right\rangle_X\right).$$

Since $\|w_k\|_{Y\cap W}$ is bounded and w_k converges to w^* strongly in X, we see the right-hand side of the above equality converges to $-\mathrm{Im}(\langle \Lambda w^*, w^*\rangle_X)$ as k goes to ∞. The claim is proved. We set $\mu^* = \lim_{k\to\infty} \frac{\mathrm{Im}(\mu_k)}{\alpha_k}$.

Let $f \in D(\mathcal{L})$. Then we have

$$\begin{aligned}
i\mu^* \langle w^*, f\rangle_X &= \lim_{k\to\infty} \frac{\mu_k}{\alpha_k}\langle w_k, f\rangle_X \\
&= \lim_{k\to\infty} \frac{1}{\alpha_k}\langle L_{\alpha_k,\lambda} w_{\alpha_k}, f\rangle_X \\
&= \lim_{k\to\infty} \left(\frac{1}{\alpha_k}\left\langle (\mathcal{L} + \lambda\mathcal{M}) w_k, f \right\rangle_X - \langle \Lambda w_k, f\rangle_X - \frac{1}{\alpha_k}\left\langle (\lambda\Lambda_{w_\infty} + \Lambda_{w_{\alpha_k,\lambda}}) w_k, f \right\rangle_X \right) \\
&= -\langle \Lambda w^*, f\rangle_X.
\end{aligned}$$

Since $D(\mathcal{L})$ is dense in X, we get $\Lambda w^* = -i\mu^* w^*$, that is, μ^* is an eigenvalue of Λ and w^* is the associated eigenfunction with $\|w^*\|_X = 1$. However, from [12, Lemma 5.1], Λ has no eigenvalues except for the eigenvalue zero. Thus $\mu^* = 0$, and from Lemma 2.2, we see that w^* is a radially symmetric function (note that w^* belongs to X_1). This leads to

$$\lim_{k\to\infty} \langle \Lambda_{w_\infty} w_k, w_k\rangle_X = \langle \Lambda_{w_\infty} w^*, w^*\rangle_X = 0,$$

since $\Lambda_{w_\infty} w^* \in \mathbb{P}_S^\perp X$ which follows from the fact that w^* is radially symmetric; see Corollary 2.2. By (6.1) we also have

$$\lim_{k\to\infty} \langle \Lambda_{w_{\alpha_k,\lambda}} w_k, w_k\rangle_X = 0.$$

Hence we have

$$\begin{aligned}
-1 &< \liminf_{k\to\infty} \mathrm{Re}(\mu_k) \\
&= \liminf_{k\to\infty} \mathrm{Re}\big(\langle L_{\alpha_k,\lambda} w_k, w_k\rangle_X\big) \\
&= \liminf_{k\to\infty} \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w_k, w_k\big\rangle_X\big) - \liminf_{k\to\infty} \mathrm{Re}\big(\big\langle(\lambda\Lambda_{w_\infty} + \Lambda_{w_{\alpha_k,\lambda}})w_k, w_k\big\rangle_X\big) \\
&= \liminf_{k\to\infty} \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w_k, w_k\big\rangle_X\big).
\end{aligned}$$

Let us calculate $\liminf_{k\to\infty} \mathrm{Re}(\langle(\mathcal{L}+\lambda\mathcal{M})w_k, w_k\rangle_X)$. We have

$$\begin{aligned}
\mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w_k, w_k\big\rangle_X\big) &= \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})(w_k-w^*), w_k-w^*\big\rangle_X\big) + \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w^*, w_k-w^*\big\rangle_X\big) \\
&\quad + \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})(w_k-w^*), w^*\big\rangle_X\big) + \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w^*, w^*\big\rangle_X\big).
\end{aligned}$$

For the first term of the right-hand side, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})(w_k-w^*), w_k-w^*\big\rangle_X\big) \\
&\quad = -\liminf_{k\to\infty} \int_{\mathbb{R}^2} \left\{ \left|\nabla\left(G^{-\frac{1}{2}}(w_k-w^*)\right)\right|^2 + G^{-1}\frac{|x|^2}{16}|w_k-w^*|^2 - \frac{1}{2}G^{-1}|w_k-w^*|^2 + \frac{\lambda}{8}G^{-1}\big(x_1^2-x_2^2\big)|w_k-w^*|^2 \right\} dx \\
&\quad \leqslant \liminf_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}^2} G^{-1}|w_k-w^*|^2\,dx \\
&\quad = 0.
\end{aligned}$$

Since w_k converges to w^* weakly in $Y \cap W$, we also have

$$\liminf_{k\to\infty} \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w^*, w_k-w^*\big\rangle_X\big) + \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})(w_k-w^*), w^*\big\rangle_X\big) = 0.$$

Finally, we calculate $\mathrm{Re}(\langle(\mathcal{L}+\lambda\mathcal{M})w^*, w^*\rangle_X)$. It is easy to check that $\langle\mathcal{M}w^*, w^*\rangle_X = 0$ since w^* is radially symmetric. Moreover, from the radial symmetry, w^* belongs to X_1 in which we have $-\mathcal{L} \geqslant 1$. Hence we obtain

$$\mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w^*, w^*\big\rangle_X\big) = \mathrm{Re}\big(\langle\mathcal{L}w^*, w^*\rangle_X\big) \leqslant -\|w^*\|_X^2 = -1.$$

Collecting these above, we have

$$-1 < \liminf_{k\to\infty} \mathrm{Re}(\mu_k) \leqslant \mathrm{Re}\big(\big\langle(\mathcal{L}+\lambda\mathcal{M})w^*, w^*\big\rangle_X\big) \leqslant -1,$$

which is a contradiction. The proof of the lemma is completed. □

Similar calculations as in (6.6) give the following proposition.

Proposition 6.1. *There is a real number γ such that the resolvent $\rho(L_{\alpha,\lambda})$ contains a half plane $\{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \geqslant \gamma\}$ and*

$$\big\|(L_{\alpha,\lambda} - \mu I)^{-1}\big\|_{X\to X} \leqslant \frac{C}{|\mu|} \tag{6.8}$$

for any $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geqslant \gamma$. Here C depends only on α and λ.

We omit the proof here. The important consequence of this proposition is that $L_{\alpha,\lambda}$ generates the analytic semigroup $e^{tL_{\alpha,\lambda}}$ in X_1; see [10, Proposition 2.1.11]. Let $r_{\alpha,\lambda}$ be the growth bound of the semigroup $e^{tL_{\alpha,\lambda}}$ in X_1, that is,

$$r_{\alpha,\lambda} := \sup\bigl\{r \in \mathbb{R} \bigm| \text{there is an } M_r \geqslant 1 \text{ such that } \bigl\|e^{tL_{\alpha,\lambda}}\bigr\|_{X\to X} \leqslant M_r e^{rt},\ t>0\bigr\}. \tag{6.9}$$

Since $e^{tL_{\alpha,\lambda}}$ is an analytic semigroup, $r_{\alpha,\lambda}$ coincides with the spectral bound $\sup_{\mu\in\sigma(L_{\alpha,\lambda})} \mathrm{Re}(\mu)$. Hence, from Lemma 6.1 we have

Proposition 6.2. *For any $\eta \in (0,1)$ there is an $R(\eta) \gg 1$ such that if $|\alpha| \geqslant R(\eta)$ then for all $f \in X_1$, the estimate*

$$\bigl\|e^{tL_{\alpha,\lambda}} f\bigr\|_X \leqslant Ce^{-\eta t}\|f\|_X, \quad t>0 \tag{6.10}$$

holds. Here the constant C depends only on α, λ, and η.

In order to get the local stability of $\omega_{\alpha,\lambda}$ we also need the estimates for the first derivatives with respect to spatial variables for $e^{tL_{\alpha,\lambda}} f$ because of the nonlinear term in the original equations (6.2). In fact, we can show the estimate

$$\bigl\|\nabla e^{tL_{\alpha,\lambda}} f\bigr\|_X \leqslant \frac{C(1+t^{\frac12})}{t^{\frac12}} e^{-\eta t}\|f\|_X, \quad t>0, \tag{6.11}$$

for $\eta \in (0,1)$ and $|\alpha| \geqslant R(\eta) \gg 1$. We omit the details of the proof for (6.11), but one way to establish this estimate is to solve the integral equations

$$w(t) = e^{t(\mathcal{L}+\lambda\mathcal{M})} f - \int_0^t e^{(t-s)(\mathcal{L}+\lambda\mathcal{M})} \Lambda_{\omega_{\alpha,\lambda}} w(s)\,ds \tag{6.12}$$

where $e^{t(\mathcal{L}+\lambda\mathcal{M})}$ is the semigroup generated by $\mathcal{L}+\lambda\mathcal{M}$ in X_1. Clearly $w(t) = e^{tL_{\alpha,\lambda}} f$. By using the explicit representation of $e^{t(\mathcal{L}+\lambda\mathcal{M})}$ (see [5])

$$e^{t(\mathcal{L}+\lambda\mathcal{M})} = \frac{e^t}{4\pi(\lambda_1(t)\lambda_2(t))^{\frac12}} \int_{\mathbb{R}^2} e^{-\frac{(x_1-y_1)^2}{4\lambda_1(t)} - \frac{(x_2-y_2)^2}{4\lambda_2(t)}} f\bigl(y_1 e^{\frac{(1+\lambda)t}{2}}, y_2 e^{\frac{(1-\lambda)t}{2}}\bigr)\,dy \tag{6.13}$$

where $\lambda_1(t) = \frac{1-e^{-(1+\lambda)t}}{1+\lambda}$ and $\lambda_2(t) = \frac{1-e^{-(1-\lambda)t}}{1-\lambda}$, one can derive the estimate

$$\bigl\|\nabla e^{t(\mathcal{L}+\lambda\mathcal{M})} f\bigr\|_X \leqslant Ct^{-\frac12}\|f\|_X \quad \text{for } t \ll 1,$$

which leads to (6.11) for $t \ll 1$ by solving (6.12). Then by the semigroup property and (6.10), we get (6.11) for all $t>0$. Note that the constant C in (6.11) depends only on α, λ, and η.

Solutions of (6.2) are given as solutions of the integral equations

$$w(t) = e^{tL_{\alpha,\lambda}} w_0 - \int_0^t e^{(t-s)L_{\alpha,\lambda}} (K*w,\nabla) w(s)\,ds. \tag{6.14}$$

From the Gagliardo–Nirenberg inequality, we have the estimate of the nonlinear term $(K*w,\nabla)w$ such as

$$\begin{aligned}
\bigl\|(K*w,\nabla)w\bigr\|_X &\leqslant C\|K*w\|_{L^\infty}\|\nabla w\|_X \\
&\leqslant C\|w\|_{L^p}^{\frac14}\|w\|_{L^r}^{\frac34}\|\nabla w\|_X \\
&\leqslant C\|w\|_{L^p}^{\frac14}\|w\|_{L^2}^{\frac34(1-\sigma)}\|\nabla w\|_{L^2}^{\frac34\sigma}\|\nabla w\|_X,
\end{aligned}$$

where $2<r<3$, $p = \frac{r}{2r-3} < 2$, and $\sigma = 1-\frac{2}{r}$. Hence, we see

$$\bigl\|(K*w,\nabla)w\bigr\|_X \leqslant C\|w\|_X^{1-\frac{3\sigma}{4}}\|\nabla w\|_X^{1+\frac{3\sigma}{4}}. \tag{6.15}$$

Then for $\eta \in (0,1)$ and $|\alpha| \geqslant R(\eta) \gg 1$ we can solve (6.14) in the closed ball

$$\Bigl\{f \in X_1 \Bigm| \sup_{t>0} e^{\eta t}\bigl\|f(t)\bigr\|_X + \sup_{t>0} \frac{t^{\frac12}}{(1+t^{\frac12})} e^{\eta t}\bigl\|\nabla f(t)\bigr\|_X \leqslant \epsilon\Bigr\}, \tag{6.16}$$

if $\epsilon > 0$ is sufficiently small depending on α, λ, and η. Since this argument is well known, we omit the details. Theorem 1.3 is now proved.

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